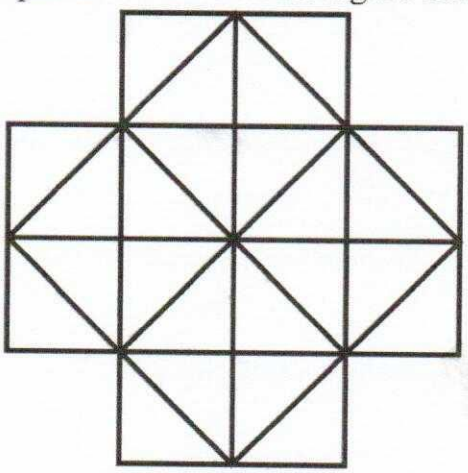
5th Challenge for Future Mathematicians
Bogor, Oktober 27-30, 2018
Junior High School Category
Team Contest-**SOLUTION**
Time : 80 minutes

1. How many triangles and squares are there in the figure below? (Vietnam)

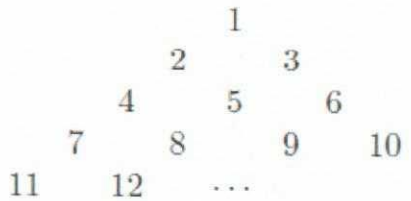


Ans: 52 triangles; 22 squares.
Hint: Consider that each small triangle is of size 1.
Calculate the number of triangles of size 1, size 2, size 4, size 8: 24, 16, 8, 4
Calculate the number of square of size 2, 4, 8, 16: 12, 4, 5, 1

Proposed Marking Scheme:

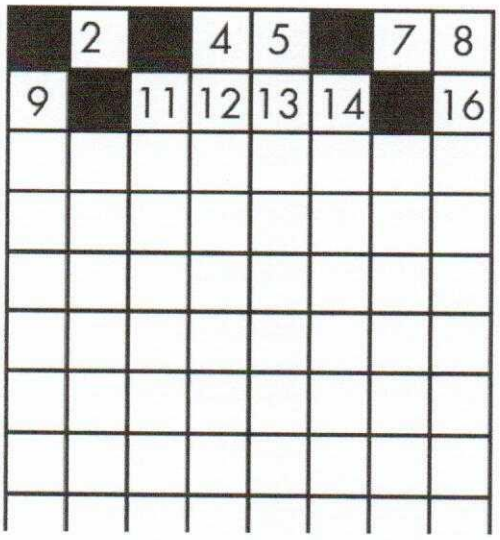
- Correct number of triangles **(20 points)**
- Correct number of squares **(20 points)**

2. The positive integers are arranged in increasing order in a triangle, as shown. Each row contains one more number than the previous row. The sum of the numbers in the row that contains the number 500 is ….
(Malaysia)



The largest number in the n-th row is
$T = 1 + 2 + 3 + \_\_\_ + (n-2) + (n-1) + n$
This expression is always equal to $\frac{1}{2}n(n+1)$.
i.e the largest number in the nth row is $\frac{1}{2}n(n+1) \geq 500$ or $n(n+1) \geq 1000$
If $n = 31$, then $n(n+1) = 992$. Last $T = 496$
If $n = 32$, then $n(n+1) = 1056$. Last $T = 528$
$\therefore 32^{nd}$ row: 497, 498, . . . , 527, 528
Sum = ½ (528 × 529) − ½ (496 × 497) = 139656 − 123256 = 16400
or $N = 528 - 597 + 1 = 32$; $S_N = 32/2 \times (497 + 528) = 16 \times 1025 = 16400$

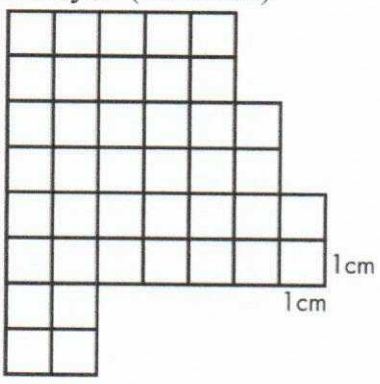
3. Given a table of 8 columns which are numbered from left to right and from top to bottom with the consecutive natural numbers starting from 1. Then, we color the first cell (cell number 1) in black, keep the next cell the same (cell number 2), color the third cell (cell number 3) in black; keep the next 2 cells (cells number 4 and 5) the same; color the next cell (cell number 6) in black; keep the next 3 cells (cells numbers 7, 8, 9) the same; color the next cell (cell number 10) in black, …, and so on. From the cell number 1 to the cell number 2018, how many white cells are there?
(Vietnam)



Ans: 1955 white cells

*Hint: The number in the nth black cell is: 1 + 2 + ... + n = n(n+1)/2. From that, and 63*64/2 = 2016, one has that the 63th black cell is numbered 2016. Then in the first 2018 cells, one has 63 black cell, so the number of white cells are 1955.*

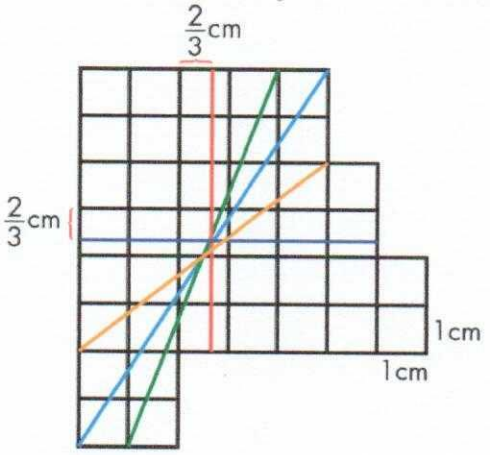
4. Use only 1 straight line to divide the figure below into 2 parts whose areas are the same. Draw 4 different ways. (Vietnam)



Proposed Marking Scheme:

- 10 points for every correct solution, need 4 correct solutions.

There are a lot of ways to do. For example, 5 ways are shown below:



5. The smallest positive integer k for which $k(k+1)(k+2)$ is a multiple of 5 is $k = 3$. All positive integers, k, for which $k(k+1)(k+2)$ is a multiple of 5 are listed in increasing order. What is the 1000th integer in the list? (Malaysia)

   $k = 3, 4, 5. 8, 9, 10 \rightarrow$ in a block of 10 integers, 6 are multiples of 5.

   Note: $1000 = 166 \times 6 + 4$. 166 block of $10 = 1660$

   This means that, in the first $166 \times 10 = 1660$ positive integers, there are $166 \times 6 = 966$ integers

   k for which $k(k+1)(k+2)$ is a multiple of 5.

   The 1000th integer ends with $8 \rightarrow 1668$.

Proposed Marking Scheme:

- Observe that there are for every 10 integers, there are 6 integers that are multiples of 5 **(5 points)**
- Deducing properly the conditions **(up to 15 points)**
- Deduced to get the correct answer **(20 points)**

6. How many 7-digit numbers are there which are multiples of 11 and whose sum of its digits is 61?

Answer: 6

Solution: Let n be a 7-digit number with digit sum 61. The possible digit combinations for n are (8,8,9,9,9,9,9) and (7,9,9,9,9,9,9). Let a be the sum of the digits of n in the odd positions and b be the sum of the digits in the even positions. Then $a+b = 61$, and $a-b$ is a multiple of 11, so $a-b = 11k$. It follows that $2a = 61+11k$. Since $34 \leq a \leq 36$, we must have $k=1$ and $a=36$, so that the digits in the odd positions are all 9's. Thus, there are 6 numbers with the desired properties namely, 9998989, 9899989, 9898999, 9999979, 9997999 and 9799999.

(Philippines)

Proposed Marking Scheme:

- Get all 6 numbers without mistake (40 points)
- Get 5 numbers without mistake (30 points)
- Get 4 numbers without mistake (20 points)
- Get 3 numbers without mistake (10 points)
- Get 1 or 2 numbers without mistake (0 points)

Note: for every mistake given, we deduct 5 points.

7. Consider the set of all 9- digit numbers that are made up of all digits 1, 2, 3, ..., 9. If you list the numbers of the set in increasing order and divide the list into four parts (the number of numbers in each part is equal), what is the last number of the first part? (Vietnam)

Ans: 329876541

Proposed Marking Scheme:

The set has a total of 9! numbers. **(5 points)**

If we divide the list into four part of the same size, each set has 9!/4 numbers. **(5 points)**

So, the first digit of last number of the first part is 3. **(5 points)**

The number of numbers where the first digit is either 1 or 2 is: 2 x 8! numbers **(5 points)**

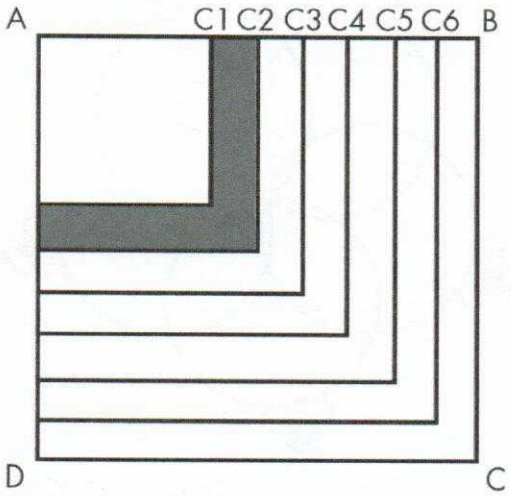
So the number of numbers whose the first digit is 3 is:

9!/4 - 2x8! = (18 - 16) x 7! = 2 x 7! **(5 points)**

So, the last number of the first part is: 329876541 **(15 points)**

Note: If correct answer only, with no solutions, give 5 points.

8. Squares whose side lengths are natural numbers are put on each other in the decreasing order of their side lengths, the square whose side length is the smallest is on the top. The sides C_1C_2, C_2C_3, C_3C_4, C_4C_5, C_5C_6 and C_6B are equal. The area of the gray part is 19. What is the area of the square ABCD? (Vietnam)



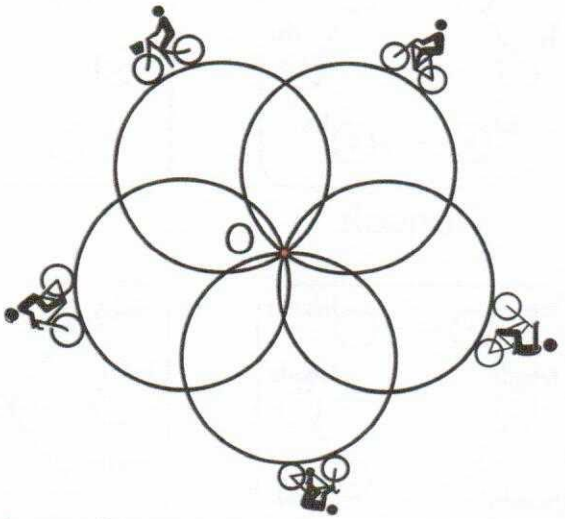
Ans: 225

Hint: The gray part is equal to the area of a squar e of size C1C2 and 2 rectangles which one size is C1C2. As all size are natural numbers so 19 is divisible by C1C2. So C1C2 is 1 and AC1 is 9. AB = 15 and we have the solution.

Proposed Marking Scheme:

- If they get the value of C1C2 = 1 (10 points)
- If they get the value of AC1 = 9 (10 points)
- Deduce that the square is a 15 x 15 square (10 points)
- Get the correct answer (10 points)

9. Five students ride their bicycles on 5 circles at the same time at 8:00 and start at the intersection point O as in the figure below. Each one rides on a different circle. Given that their constant speeds are: 8 mph, 12 mph, 16 mph, 20 mph and 24 mph. The length of each circle is $\frac{1}{2}$ miles. When will they all meet each other the fifth times?

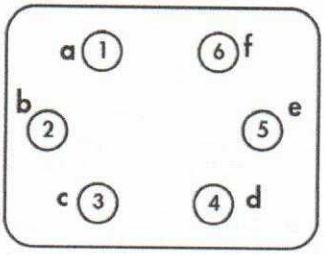


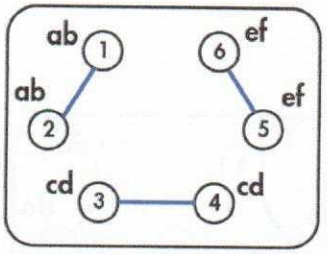
Ans: 8 hours 37 minutes 30 seconds.
Solution:

To finish 1 round, they need are $\frac{1}{16}; \frac{1}{24}; \frac{1}{32}; \frac{1}{40}; \frac{1}{48}$ hours, respectively (10 points)
The least time it takes the students to meet each other at the point O the first time is
1/LCD (16, 24, 32, 40, 48) = $\frac{1}{8}$ hour. (10 points)
The time it takes them to meet each other the fifth times is: $\frac{1}{8}$ x 5 = $\frac{5}{8}$ hour = 37 minutes 30 seconds. (10 points)

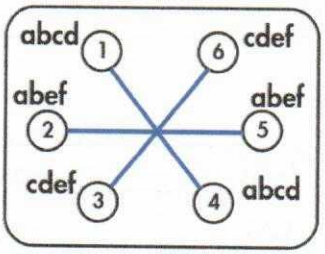
So for the fifth time, they meet after 8h 37m 30s (10 points)

10. At the weekend, 6 spies meet each other and share the information that they have in that week. They have some rounds to meet but each one will meet at most 1 person in a round. For example, a group of 6 spies only need 3 rounds to share their information as below:

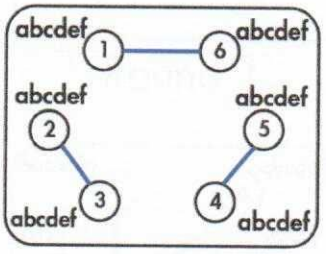




Round 1



Round 2



Round 3

- Before meeting each other, each spy number 1, 2, 3, 4, 5, 6, has their own information a, b, c, d, e, f, respectively.
- After the first round, spies 1 and 2 share their information and their new information now is "ab".
- The figure above shows who a spy will meet in a round by a line and the new information they get after meeting.
- After 3 rounds, each spy has full information of all 6 spies.

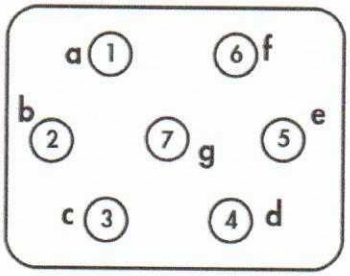
At least, how many rounds does a group of 7 spies need so that each spy has full information of all spies?

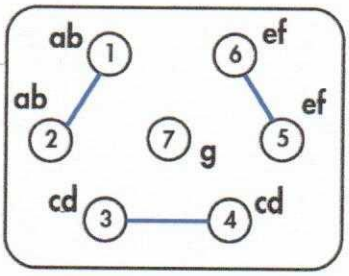
Solution:

Each spy needs 7 pieces of information. (5 points)

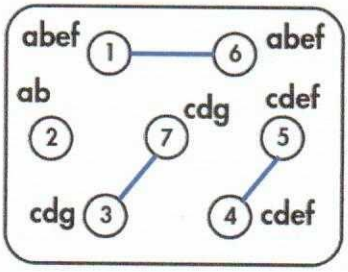
Each round, at most, there are 6 people sharing information. (5 points)

- After the 1st round, only one person has 1 piece of information. At most, each person has 2 pieces of information.
- After the 2nd round, at most each person has 4 pieces of information, only one person has 1 or 2 pieces of information - Let call him Mr X.
- After the 3rd round, Mr X has at most 6 pieces of information. (5points) So the number of rounds we need is greater than 3. (10 points) We can arrange the first 3 rounds as follows:

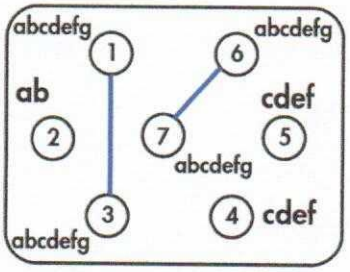




Round 1



Round 2



Round 3

After these rounds, 4 spies 1,3,6,7 have all 7 pieces of information. In the round 4, the spies 2, 4, 5 can meet the 3 spies which have all 7 pieces of information to complete their information.

So we need at least 4 rounds. (15 points - with one way of exchanging).